

QUASIAFFINE ORBITS OF INVARIANT SUBSPACES FOR UNIFORM JORDAN OPERATORS

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ABSTRACT. For a uniform Jordan operator T and two invariant subspaces M_1 and M_2 , we show that M_2 belongs to the quasiaffine orbit of M_1 if and only if the restrictions $T|_{M_1}$ and $T|_{M_2}$ are quasisimilar and the compression $T_{M_2^\perp}$ can be injected in the compression $T_{M_1^\perp}$.

1. INTRODUCTION

Let $T_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ and $T_2 : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ be bounded linear operators on Hilbert spaces. If M_1 and M_2 are invariant subspaces for T_1 and T_2 respectively (that is $M_1 \subset \mathcal{H}_1$ and $M_2 \subset \mathcal{H}_2$ are closed subspaces such that $T_1 M_1 \subset M_1$ and $T_2 M_2 \subset M_2$), we say that M_2 is a *quasiaffine transform* of M_1 if there exists a bounded injective operator with dense range $X : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $XT_1 = T_2 X$ and $\overline{XM_1} = M_2$. We write $M_1 \prec M_2$ when M_2 is a quasiaffine transform of M_1 . In that case, we also say that M_2 lies in the *quasiaffine orbit* of M_1 . When $M_1 \prec M_2$ and $M_2 \prec M_1$, we say that M_1 and M_2 are *quasisimilar* and write $M_1 \sim M_2$. Quasisimilarity is clearly an equivalence relation on the class of pairs of the form (T, M) , where M is an invariant subspace for the bounded linear operator T . In [2], Bercovici raised the basic problem underlying our present investigation: describe the quasiaffine orbit of a given invariant subspace for an operator of class C_0 (see definition in Section 2).

Related results for general operators of class C_0 can be found in [2], where it is proved that the quasisimilarity class of an invariant subspace is determined by the quasisimilarity class of the restriction $T|M$ if and only if T has property (Q). Nilpotent operators of finite multiplicity have been considered in [6]. In that context, it was proved that the quasisimilarity class of M is determined by the quasisimilarity classes of the restriction $T|M$ and of the compression T_{M^\perp} when either of those operators has multiplicity one. In fact, for any operator T of class C_0 with the property that $T|M$ has multiplicity one, the weakly quasiaffine orbit of the invariant subspace is determined by the quasisimilarity classes of $T|M$ and T_{M^\perp} (see [4]).

The objects we will be concerned with in this work are the so-called uniform Jordan operators (that is $T = S(\theta) \oplus S(\theta) \oplus \dots$). These operators appear to be more amenable, and our understanding of the quasisimilarity classes of their invariant subspaces is significantly better. In their pioneer work (see [5]), Bercovici and Tannenbaum considered the case where T has finite multiplicity and established that $M_1 \sim M_2$ if and only if $T|M_1 \sim T|M_2$. Moreover, it was observed that for $T = S(z^2) \oplus S(z)$, this classification breaks down, so the corresponding result fails if T is not uniform. Later on, it was proved in [2] that this classification holds for a uniform Jordan operator T if and only if $T|M$ satisfies property (P). In general, the quasisimilarity class of an invariant subspace for a uniform Jordan operator is determined by the quasisimilarity classes of the restriction $T|M$ and of the compression T_{M^\perp} (see [3]). In this paper, we focus on the weaker notion of quasiaffine orbit. The main theorem gives a characterization of these orbits for uniform Jordan operators and as such it extends the aforementioned result.

2. BACKGROUND AND PRELIMINARIES

We give here some background concerning operators of class C_0 . Let H^∞ be the algebra of bounded holomorphic functions on the open unit disc \mathbb{D} . Let \mathcal{H} be a Hilbert space and T a bounded linear operator

on \mathcal{H} , which we indicate by $T \in \mathcal{B}(\mathcal{H})$. The operator T is said to be of *class* C_0 if there exists an algebra homomorphism $\Phi : H^\infty \rightarrow \mathcal{B}(\mathcal{H})$ with the following properties:

- (i) $\|\Phi(u)\| \leq \|u\|$ for every $u \in H^\infty$
- (ii) $\Phi(p) = p(T)$ for every polynomial p
- (iii) Φ is continuous when H^∞ and $\mathcal{B}(\mathcal{H})$ are given their respective weak-star topologies
- (iv) Φ has non-trivial kernel.

We use the notation $\Phi(u) = u(T)$, which is the Sz.-Nagy–Foias H^∞ functional calculus. It is known that $\ker \Phi = m_T H^\infty$ for some inner function m_T called the *minimal function* of T . The minimal function is uniquely determined up to a scalar factor of absolute value one. A set $E \subset \mathcal{H}$ is said to be *cyclic* for T if $\mathcal{H} = \bigvee_{n=0}^\infty T^n E$. The *multiplicity* of the operator T is the smallest cardinality of a cyclic set. If T has multiplicity one, it is said to be *multiplicity-free*.

Let H^2 denote the Hilbert space of functions $f(z) = \sum_{n=0}^\infty a_n z^n$ holomorphic in \mathbb{D} equipped with the norm $\|f\|^2 = \sum_{n=0}^\infty |a_n|^2$. For any inner function $\theta \in H^\infty$, the space $H(\theta) = H^2 \ominus \theta H^2$ is invariant for S^* , the adjoint of the shift operator S on H^2 . The operator $S(\theta)$ defined by $S(\theta)^* = S^*|(H^2 \ominus \theta H^2)$ is called a *Jordan block*; it is of class C_0 with minimal function θ . We state some useful properties of these operators. Given functions $u, v \in H^\infty$, we say that u *divides* v and write $u|v$ if there exists a function $w \in H^\infty$ such that $v = uw$.

Proposition 2.1 ([1] Proposition 3.1.10). *Let $\theta \in H^\infty$ be an inner function.*

- (i) *The operator $S(\theta)$ is multiplicity-free.*
- (ii) *If $\phi \in H^\infty$ is an inner divisor of θ , then $\phi H^2 \ominus \theta H^2$ is an invariant subspace for $S(\theta)$. In fact,*

$$\phi H^2 \ominus \theta H^2 = \text{ran } \phi(S(\theta)) = \ker(\theta/\phi)(S(\theta)).$$

Conversely, any invariant subspace for $S(\theta)$ is of this form.

A more general family of operators are the so-called *Jordan operators*. We will define them here in the case where the Hilbert space on which they act is separable. These operators are of the form $\bigoplus_{n=0}^\infty S(\theta_n)$ where $\{\theta_n\}_{n=0}^\infty$ is a sequence of inner functions satisfying $\theta_{n+1}|\theta_n$ for $n \geq 0$. In case where $\theta_n = \theta$ for every $n \geq 0$ for some fixed inner function $\theta \in H^\infty$, then the operator $T = \bigoplus_{n=0}^\infty S(\theta)$ is called a *uniform Jordan operator*.

Recall that a bounded injective linear operator with dense range is called a *quasiaffinity*. Two operators $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$ are said to be *quasimilar* if there exist quasiaffinities $X : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and $Y : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ such that $XT_1 = T_2X$ and $T_1Y = YT_2$. We use the notation $T_1 \sim T_2$ to indicate that T_1 and T_2 are quasimilar. The Jordan operators are of fundamental importance in the study of operators of class C_0 as the following theorem illustrates.

Theorem 2.2 ([1] Theorem 3.5.1). *For any operator T of class C_0 acting on a separable Hilbert space there exists a unique Jordan operator J such that T and J are quasimilar.*

The operator J in the previous theorem is called the *Jordan model* of T . We now collect some facts about invariant subspaces for operators of class C_0 .

Proposition 2.3 ([1] Theorem 3.2.13, Theorem 3.3.8). *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator of class C_0 . The following statements are equivalent:*

- (i) *T is multiplicity-free*
- (ii) *$\{T\}'$ is commutative*
- (iii) *For every inner divisor θ of m_T there exists a unique invariant subspace $K \subset \mathcal{H}$ for T such that $m_T|_K \equiv \theta$. In fact, $K = \ker \theta(T) = \text{ran}(m_T/\theta)(T)$.*

Given a subset $E \subset \mathcal{B}(\mathcal{H})$, we denote its commutant by

$$E' = \{X \in \mathcal{B}(\mathcal{H}) : XT = TX \text{ for every } T \in E\}.$$

We denote by $\text{Lat}(T)$ the collection of invariant subspaces for an operator T , and by $\text{Alg Lat}(T)$ the algebra of operators X such that $XM \subset M$ for every $M \in \text{Lat}(T)$.

Theorem 2.4 ([1] Theorem 4.1.2). *For an operator T of class C_0 , we have $\text{Alg Lat}(T) \cap \{T\}' = \{T\}''$.*

Let us recall a relation which is weaker than that of quasisimilarity. Given $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$, we say that T_1 can be *injected* in T_2 if there exists an injective operator $X : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $XT_1 = T_2X$. We indicate the fact that T_1 can be injected in T_2 by $T_1 \prec^i T_2$. If in addition X has dense range, we say that T_2 is a *quasiaffine transform* of T_1 and we write $T_1 \prec T_2$.

Theorem 2.5 ([1] Proposition 3.5.31, Proposition 3.5.32). *Let T_1 and T_2 be two operators of class C_0 . Then, the following are equivalent:*

- (i) $T_1 \prec T_2$
- (ii) $T_1 \prec^i T_2$ and $T_2 \prec^i T_1$
- (iii) $T_1 \sim T_2$.

Moreover, $T_1 \prec^i T_2$ if and only if $T_1^* \prec^i T_2^*$. If $\bigoplus_{n=0}^{\infty} S(\theta_n^{(1)})$ and $\bigoplus_{n=0}^{\infty} S(\theta_n^{(2)})$ are the Jordan models of T_1 and T_2 respectively, then $T_1 \prec^i T_2$ if and only if $\theta_n^{(1)} | \theta_n^{(2)}$ for every $n \geq 0$.

Given an invariant subspace M for an operator T , we denote by T_{M^\perp} the compression $P_{M^\perp}T|_{M^\perp}$. The following two results concerning uniform Jordan operators are from [3].

Proposition 2.6 ([3] Proposition 2.1). *Let $T = \bigoplus_{n=0}^{\infty} S(\theta)$ and M be an invariant subspace for T . Assume that $\bigoplus_{n=0}^{\infty} S(\phi_n)$ and $\bigoplus_{n=0}^{\infty} S(\psi_n)$ are the Jordan models of $T|_M$ and T_{M^\perp} respectively. Then,*

- (i) ϕ_0 and ψ_0 divide θ
- (ii) θ divides $\phi_m \psi_n$ for every $m, n \geq 0$.

Theorem 2.7 ([3] Theorem 2.5). *Let $T = \bigoplus_{n=0}^{\infty} S(\theta)$ and M be an invariant subspace for T . Assume that $\bigoplus_{n=0}^{\infty} S(\phi_n)$ and $\bigoplus_{n=0}^{\infty} S(\psi_n)$ are the Jordan models of $T|_M$ and T_{M^\perp} respectively. Then, M is quasisimilar to*

$$\bigoplus_{n=0}^{\infty} (\gamma_n H^2 \ominus \theta H^2)$$

where $\gamma_n = \theta / \phi_{n/2}$ for n even, and $\gamma_n = \psi_{(n-1)/2}$ for n odd.

Let us close this section by proving an elementary known fact which motivates our main result.

Proposition 2.8. *Let $T_1 \in \mathcal{B}(\mathcal{H}_1), T_2 \in \mathcal{B}(\mathcal{H}_2)$ be operators of class C_0 and let $M_1 \subset \mathcal{H}_1, M_2 \subset \mathcal{H}_2$ be invariant subspaces for T_1 and T_2 respectively. Assume that $M_1 \prec M_2$. Then, $T_1|_{M_1} \sim T_2|_{M_2}$ and $(T_2)_{M_2^\perp} \prec^i (T_1)_{M_1^\perp}$.*

Proof. By assumption, there exists a quasiaffinity $X : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $XT_1 = T_2X$ and $\overline{XM_1} = M_2$. It follows that $X|_{M_1}$ implements a quasiaffine transform between $T_1|_{M_1}$ and $T_2|_{M_2}$. Hence $T_1|_{M_1} \prec T_2|_{M_2}$ and Theorem 2.5 implies that $T_1|_{M_1} \sim T_2|_{M_2}$. Moreover, we have $X^*M_2^\perp \subset M_1^\perp$ and thus $P_{M_2^\perp}XP_{M_1^\perp} = P_{M_2^\perp}X$. Since M_2 is invariant for T_2 , we have $P_{M_2^\perp}T_2P_{M_2^\perp} = P_{M_2^\perp}T_2$. If we set $A = P_{M_2^\perp}X|_{M_1^\perp}$, we see that

$$A(P_{M_1^\perp}T_1|_{M_1^\perp}) = P_{M_2^\perp}XT_1|_{M_1^\perp} = P_{M_2^\perp}T_2X|_{M_1^\perp} = (P_{M_2^\perp}T_2|_{M_2^\perp})A$$

and

$$\ker A^* = \{h \in M_2^\perp : X^*h \in M_1\} = 0.$$

Thus,

$$(P_{M_2^\perp}T_2|_{M_2^\perp})^* \prec^i (P_{M_1^\perp}T_1|_{M_1^\perp})^*.$$

By Theorem 2.5, we get

$$P_{M_2^\perp}T_2|_{M_2^\perp} \prec^i P_{M_1^\perp}T_1|_{M_1^\perp}$$

and the proof is complete. \square

Our main result shows that the converse of the previous proposition holds in case where $T_1 = T_2$ is a uniform Jordan operator.

3. UNIFORM JORDAN OPERATORS

Let us start with an elementary lemma.

Lemma 3.1. *Let $\phi, \psi \in H^\infty$ be inner divisors of the inner function $\theta \in H^\infty$. Assume that θ/ϕ divides ψ and set $\omega = \psi/(\theta/\phi)$. Then for every $g \in \psi H^2 \ominus \theta H^2$ we can find $f \in (\theta/\phi)H^2 \ominus \theta H^2$ such that $\omega(S(\theta))f = g$ and $\|f\| = \|g\|$.*

Proof. Fix $g \in \psi H^2 \ominus \theta H^2$. By Proposition 2.1, we have

$$\omega(S(\theta))((\theta/\phi)H^2 \ominus \theta H^2) = (\omega\theta/\phi)(S(\theta))(H^2 \ominus \theta H^2) = \psi H^2 \ominus \theta H^2.$$

We can thus find $f_0 \in (\theta/\phi)H^2 \ominus \theta H^2$ such that $\omega(S(\theta))f_0 = g$. We set $f = P_{H(\theta/\omega)}f_0$. Then, there exists a function $h \in H^2$ such that $f = f_0 + (\theta/\omega)h$, whence $f = f_0 + (\theta/\phi)(\theta/\psi)h \in (\theta/\phi)H^2$ and thus $f \in (\theta/\phi)H^2 \ominus \theta H^2$ since $f \in H(\theta/\omega) \subset H(\theta)$. Moreover, $\omega f \in H(\theta)$ and

$$g = \omega(S(\theta))f = P_{H(\theta)}\omega f = \omega f$$

and since ω is an inner function, we have that $\|g\| = \|f\|$. \square

The following two lemmas provide the main tool in the proof of our main result.

Lemma 3.2. *Let $\theta \in H^\infty$ be an inner function. Let $\mathcal{H} = \bigoplus_{n=0}^\infty H(\theta)$ and $T = \bigoplus_{n=0}^\infty S(\theta)$. Let $(\omega_n)_{n=0}^\infty \in H^\infty$ be a sequence of inner divisors of θ and let $\{c_n\}_{n=0}^\infty$ be a bounded sequence of positive numbers. Define*

$$X : H(\theta) \oplus \mathcal{H} \rightarrow H(\theta) \oplus \mathcal{H}$$

as follows

$$X(g \oplus (f_n)_n) = \left(g + \sum_{n=0}^\infty \frac{1}{n+1} \omega_n(S(\theta))f_n \right) \oplus (c_n f_n)_n.$$

Then, X is a quasiaffinity which commutes with $S(\theta) \oplus T$.

Proof. It is immediate that X is injective, and a routine calculation shows that X is bounded. Pick now $G \oplus (F_n)_n \in H(\theta) \oplus \mathcal{H}$. For $m \geq 0$, define

$$g_m = G - \sum_{n=0}^m \frac{1}{(n+1)c_n} \omega_n(S(\theta))F_n \in H(\theta)$$

and

$$(f_{n,m})_n = \left(\frac{1}{c_0}F_0, \frac{1}{c_1}F_1, \dots, \frac{1}{c_m}F_m, 0, \dots \right) \in \mathcal{H}.$$

We get that

$$X(g_m \oplus (f_{n,m})_n) = (G, F_1, \dots, F_m, 0, \dots)$$

and thus $\lim_{m \rightarrow \infty} X(g_m \oplus (f_{n,m})_n) = G \oplus (F_n)_n$. This shows that X has dense range. Finally, if we define $\chi \in H^\infty$ as $\chi(z) = z$ for every $z \in \mathbb{D}$, then we have

$$\begin{aligned} (S(\theta) \oplus T)X(g \oplus (f_n)_n) &= \left(S(\theta)g + \sum_{n=0}^\infty \frac{1}{n+1} S(\theta)\omega_n(S(\theta))f_n \right) \oplus (c_n S(\theta)f_n)_n \\ &= \left(S(\theta)g + \sum_{n=0}^\infty \frac{1}{n+1} (\chi\omega_n)(S(\theta))f_n \right) \oplus (c_n S(\theta)f_n)_n \\ &= \left(S(\theta)g + \sum_{n=0}^\infty \frac{1}{n+1} \omega_n(S(\theta))S(\theta)f_n \right) \oplus (c_n S(\theta)f_n)_n \\ &= X(S(\theta) \oplus T)(g \oplus (f_n)_n) \end{aligned}$$

which completes the proof. \square

Lemma 3.3. *Let $\psi_1, \psi_2 \in H^\infty$ be inner functions and $(\phi_n)_{n=0}^\infty \in H^\infty$ be a sequence of inner functions. Assume the following divisibility relations:*

- (i) ψ_2 divides ψ_1
- (ii) ϕ_n divides θ for every $n \geq 0$ and ψ_1 divides θ
- (iii) ϕ_{n+1} divides ϕ_n for every $n \geq 0$
- (iv) θ/ϕ_n divides ψ_2 for every $n \geq 0$.

Let $\mathcal{H} = \bigoplus_{n=0}^{\infty} H(\theta)$ and for each $n \geq 0$ let $\omega_n = \psi_2/(\theta/\phi_n)$. Define

$$X : H(\theta) \oplus \mathcal{H} \rightarrow H(\theta) \oplus \mathcal{H}$$

$$X(g \oplus (f_n)_n) = \left(g + \sum_{n=0}^{\infty} \frac{1}{n+1} \omega_n(S(\theta)) f_n \right) \oplus (c_n f_n)_n,$$

where $\{c_n\}_{n=0}^{\infty}$ is a sequence of positive numbers satisfying

$$(1) \quad \lim_{m \rightarrow \infty} (m+1) c_m \left(\sum_{n=0}^{m-1} \frac{1}{|(n+1)c_n|^2} \right)^{1/2} = \lim_{m \rightarrow \infty} (m+1) c_m = 0.$$

Then, we have $\overline{X(N_{\psi_1} \oplus M)} = N_{\psi_2} \oplus M$, where

$$M = \bigoplus_{n=0}^{\infty} ((\theta/\phi_n)H^2 \ominus \theta H^2) \subset \mathcal{H},$$

$$N_{\psi_1} = \psi_1 H^2 \ominus \theta H^2 \subset H(\theta),$$

$$N_{\psi_2} = \psi_2 H^2 \ominus \theta H^2 \subset H(\theta).$$

Proof. First note that $N_{\psi_1} \subset N_{\psi_2}$ since ψ_2 divides ψ_1 . Moreover, it follows from Proposition 2.1 that for every $n \geq 0$

$$(2) \quad \omega_n(S(\theta)) ((\theta/\phi_n)H^2 \ominus \theta H^2) = \psi_2 H^2 \ominus \theta H^2.$$

Therefore, we have $X(N_{\psi_1} \oplus M) \subset N_{\psi_2} \oplus M$.

Let now $G \oplus (F_n)_n \in N_{\psi_2} \oplus M$, in other words $G \in \psi_2 H^2 \ominus \theta H^2$ and $F_n \in (\theta/\phi_n)H^2 \ominus \theta H^2$ for every $n \geq 0$. It follows from (2) that

$$G - \sum_{n=0}^m \frac{1}{(n+1)c_n} \omega_n(S(\theta)) F_n \in \psi_2 H^2 \ominus \theta H^2$$

for every $m \geq 0$. Consequently, if we set $h_0 = 0$, then by Lemma 3.1 we can find for every $m \geq 1$ a function $h_m \in (\theta/\phi_m)H^2 \ominus \theta H^2$ such that

$$\frac{1}{m+1} \omega_m(S(\theta)) h_m = G - \sum_{n=0}^{m-1} \frac{1}{(n+1)c_n} \omega_n(S(\theta)) F_n$$

and

$$\|h_m\| = (m+1) \left\| G - \sum_{n=0}^{m-1} \frac{1}{(n+1)c_n} \omega_n(S(\theta)) F_n \right\|.$$

Note now that $\omega_n(S(\theta))$ is a contraction for every $n \geq 0$ since $\omega_n \in H^\infty$ is an inner function. Using the Cauchy-Schwarz inequality, we get for every $m \geq 1$ that

$$\|h_m\| \leq (m+1) \left(\|G\| + \|(F_n)_n\| \left(\sum_{n=0}^{m-1} \frac{1}{|(n+1)c_n|^2} \right)^{1/2} \right).$$

Hence, we have that $c_m \|h_m\| \rightarrow 0$ as $m \rightarrow \infty$. Indeed

$$c_m \|h_m\| \leq (m+1) c_m \|G\| + (m+1) c_m \|(F_n)_n\| \left(\sum_{n=0}^{m-1} \frac{1}{|(n+1)c_n|^2} \right)^{1/2}$$

and the right-hand side goes to zero as m goes to infinity in view of (1). Moreover,

$$X \left(0 \oplus \left(\frac{1}{c_0} F_0, \dots, \frac{1}{c_{m-1}} F_{m-1}, h_m, 0, \dots \right) \right) = G \oplus (F_0, \dots, F_{m-1}, c_m h_m, 0 \dots),$$

Therefore,

$$\left\| G \oplus (F_n)_n - X \left(0 \oplus \left(\frac{1}{c_0} F_0, \dots, \frac{1}{c_{m-1}} F_{m-1}, h_m, 0, \dots \right) \right) \right\|^2 = \|F_m - c_m h_m\|^2 + \sum_{n=m+1}^{\infty} \|F_j\|^2$$

and thus

$$\lim_{m \rightarrow \infty} X \left(0 \oplus \left(\frac{1}{c_0} F_0, \dots, \frac{1}{c_{m-1}} F_{m-1}, h_m, 0, \dots \right) \right) = G \oplus (F_n)_n.$$

But

$$0 \oplus \left(\frac{1}{c_0} F_0, \dots, \frac{1}{c_{m-1}} F_{m-1}, h_m, 0, \dots \right) \in N_{\psi_1} \oplus M$$

for every $m \geq 1$, so that $\overline{X(N_{\psi_1} \oplus M)} = N_{\psi_2} \oplus M$. \square

We can now establish our main result.

Theorem 3.4. *Let $T = \bigoplus_{n=0}^{\infty} S(\theta)$ and M_1, M_2 be invariant subspaces for T . Then $M_1 \prec M_2$ if and only if $T|M_1 \sim T|M_2$ and $T_{M_2^\perp} \prec^i T_{M_1^\perp}$.*

Proof. One direction follows from Proposition 2.8. Assume therefore that $T|M_1 \sim T|M_2$ and $T_{M_2^\perp} \prec^i T_{M_1^\perp}$. Let $\bigoplus_{n=0}^{\infty} S(\phi_j)$ be the Jordan model of $T|M_1$ and $T|M_2$, and let $\bigoplus_{n=0}^{\infty} S(\psi_j)$ and $\bigoplus_{n=0}^{\infty} S(\tau_j)$ be the Jordan models of $T_{M_1^\perp}$ and $T_{M_2^\perp}$ respectively. Define

$$M'_1 = \bigoplus_{n=0}^{\infty} \gamma_n H^2 \ominus \theta H^2,$$

$$M'_2 = \bigoplus_{n=0}^{\infty} \delta_n H^2 \ominus \theta H^2$$

where $\gamma_n = \theta/\phi_{n/2}$ for n even, $\gamma_n = \psi_{(n-1)/2}$ for n odd, $\delta_n = \theta/\phi_{n/2}$ for n even and $\delta_n = \tau_{(n-1)/2}$ for n odd. By Theorem 2.7, we have that $M_1 \sim M'_1$ and $M_2 \sim M'_2$, so it suffices to show that $M'_1 \prec M'_2$.

Let $\mathcal{H} = \bigoplus_{n=0}^{\infty} H(\theta)$ and fix a bijection $\Phi : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$. Define

$$V : \mathcal{H} \rightarrow \bigoplus_{n=0}^{\infty} (H(\theta) \oplus \mathcal{H})$$

as follows

$$V(f_0, g_0, f_1, g_1, \dots) = \bigoplus_{n=0}^{\infty} (g_n \oplus (f_{\Phi^{-1}(n,m)})_{m=0}^{\infty}).$$

It is clear that V is an isometry with isometric inverse being

$$\bigoplus_{n=0}^{\infty} (g_n \oplus (f_{n,m})_{m=0}^{\infty}) \mapsto (f_{\Phi(0)}, g_0, f_{\Phi(1)}, g_1, \dots).$$

Hence, V is unitary and a straightforward verification shows that V satisfies

$$VT = \left(\bigoplus_{n=0}^{\infty} (S(\theta) \oplus T) \right) V.$$

Abusing notation, we will identify $\Phi^{-1}(n, m)$ with (n, m) for every $n, m \geq 0$. We have

$$VM'_1 = \bigoplus_{n=0}^{\infty} \left((\psi_n H^2 \ominus \theta H^2) \oplus \bigoplus_{m=0}^{\infty} \left(\frac{\theta}{\phi_{n,m}} H^2 \ominus \theta H^2 \right) \right)$$

and similarly

$$VM'_2 = \bigoplus_{n=0}^{\infty} \left((\tau_n H^2 \ominus \theta H^2) \oplus \bigoplus_{m=0}^{\infty} \left(\frac{\theta}{\phi_{n,m}} H^2 \ominus \theta H^2 \right) \right).$$

Note now that since $T_{M_2^\perp} \prec^i T_{M_1^\perp}$, Theorem 2.5 implies that τ_n divides ψ_n for every $n \geq 0$. We see by Proposition 2.6 that we can apply Lemma 3.2 and Lemma 3.3 to get for each $n \geq 0$ a quasiaffinity X_n commuting with $S(\theta) \oplus T$ and satisfying

$$\overline{X_n \left((\psi_n H^2 \ominus \theta H^2) \oplus \bigoplus_{m=0}^{\infty} \left(\frac{\theta}{\phi_{n,m}} H^2 \ominus \theta H^2 \right) \right)} = (\tau_n H^2 \ominus \theta H^2) \oplus \bigoplus_{m=0}^{\infty} \left(\frac{\theta}{\phi_{n,m}} H^2 \ominus \theta H^2 \right).$$

Set

$$Y = V^* \left(\bigoplus_{n=0}^{\infty} \frac{X_n}{\|X_n\|} \right) V.$$

It is then easy to check that $\overline{YM'_1} = M'_2$ and that Y is a quasiaffinity commuting with T . Hence, $M'_1 \prec M'_2$ and we are done. \square

4. UNIFORM JORDAN MODELS

The aim of this section is to relax the assumption on T being a uniform Jordan operator. Namely, we would like to get a result analogous to Theorem 3.4 in the case where T is merely quasisimilar to a uniform Jordan operator. We can actually achieve this under an extra assumption. We first need a preliminary fact.

Lemma 4.1. *Let T be an operator of class C_0 and let $X \in \text{Alg Lat}(T) \cap \{T\}'$ be an injective operator. Then, $\overline{XM} = M$ for every $M \in \text{Lat}(T)$.*

Proof. Let $M \in \text{Lat}(T)$. Using Theorem 2.2, we can decompose M into cyclic subspaces: $M = \bigvee_{j=0}^{\infty} K_j$ where $K_j \in \text{Lat}(T)$ and $T|_{K_j}$ is multiplicity-free. Since $X \in \text{Alg Lat}(T)$, we have $\overline{XK_j} \subset K_j$ for every $j \geq 0$. On the other hand, the fact that X is an injective operator commuting with T implies that $T|_{K_j} \sim T|_{\overline{XK_j}}$ for every $j \geq 0$. By Proposition 2.3, we conclude that $\overline{XK_j} = K_j$ for every $j \geq 0$, which in turn implies $\overline{XM} = \bigvee_{j=0}^{\infty} \overline{XK_j} = \bigvee_{j=0}^{\infty} K_j = M$. \square

Theorem 4.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator of class C_0 with uniform Jordan model $J = \bigoplus_{n=0}^{\infty} S(\theta)$. Assume that we can find quasiaffinities $X : \mathcal{H} \rightarrow \bigoplus_{n=0}^{\infty} H(\theta)$ and $Y : \bigoplus_{n=0}^{\infty} H(\theta) \rightarrow \mathcal{H}$ with the property that $XT = JX$, $YJ = TY$ and $YX \in \text{Alg Lat}(T)$. Then, given $M_1, M_2 \in \text{Lat}(T)$ we have that $M_1 \prec M_2$ if and only if $T|_{M_1} \sim T|_{M_2}$ and $T_{M_2^\perp} \prec^i T_{M_1^\perp}$.*

Proof. One direction follows from Proposition 2.8. Assume therefore that $T|_{M_1} \sim T|_{M_2}$ and $T_{M_2^\perp} \prec^i T_{M_1^\perp}$. Let $E_1 = \overline{XM_1}$ and $E_2 = \overline{XM_2}$. It follows that $J|_{E_1} \sim T|_{M_1} \sim T|_{M_2} \sim J|_{E_2}$. Moreover, notice that YX is a quasiaffinity commuting with T , so that by Lemma 4.1 we have $\overline{YXM_k} = M_k$ for $k = 1, 2$. In particular, this shows that $\overline{YE_k} = M_k$ for $k = 1, 2$. We can therefore write $X^*E_k^\perp \subset M_k^\perp$ and $Y^*M_k^\perp \subset E_k^\perp$ for $k = 1, 2$. Since $T_{M_2^\perp}$ can be injected into $T_{M_1^\perp}$, it follows from Theorem 2.5 that $(T_{M_2^\perp})^* = T^*|_{M_2^\perp}$ can be injected into $(T_{M_1^\perp})^* = T^*|_{M_1^\perp}$, so we can find an injective operator $Z : M_2^\perp \rightarrow M_1^\perp$ such that $Z(T^*|_{M_2^\perp}) = (T^*|_{M_1^\perp})Z$. Set $W = Y^*ZX^*|_{E_2^\perp} : E_2^\perp \rightarrow E_1^\perp$, which is obviously injective. Note that for $k = 1, 2$ we have

$$(X^*|_{E_k^\perp})(J^*|_{E_k^\perp}) = X^*J^*|_{E_k^\perp} = T^*X^*|_{E_k^\perp} = (T^*|_{M_k^\perp})(X^*|_{E_k^\perp}),$$

$$(Y^*|_{M_k^\perp})(T^*|_{M_k^\perp}) = Y^*T^*|_{M_k^\perp} = J^*Y^*|_{M_k^\perp} = (J^*|_{E_k^\perp})(Y^*|_{M_k^\perp}),$$

and it follows that

$$\begin{aligned}
W(J^*|E_2^\perp) &= Y^*ZX^*J^*|E_2^\perp \\
&= Y^*ZT^*X^*|E_2^\perp \\
&= Y^*Z(T^*|M_2^\perp)X^*|E_2^\perp \\
&= Y^*(T^*|M_1^\perp)ZX^*|E_2^\perp \\
&= (J^*Y^*|M_1^\perp)ZX^*|E_2^\perp \\
&= (J^*|E_1^\perp)Y^*ZX^*|E_2^\perp \\
&= (J^*|E_1^\perp)W.
\end{aligned}$$

Thus $J^*|E_2^\perp$ can be injected into $J^*|E_1^\perp$, whence $J_{E_2^\perp}$ can be injected into $J_{E_1^\perp}$ via another application of Theorem 2.5. By Theorem 3.4, we can find a quasiaffinity $A \in \{J\}'$ such that $\overline{AE_1} = E_2$. Define finally $B = YAX$ which is clearly another quasiaffinity. We then have

$$BT = YAXT = YAJX = YJAX = TYAX = TB$$

and

$$\overline{BM_1} = \overline{YAXM_1} = \overline{YAE_1} = \overline{YE_2} = M_2$$

and the proof is complete. \square

In closing, let us mention an example where the extra assumption $YX \in \text{Alg Lat}(T)$ appearing in Theorem 4.2 is satisfied.

Corollary 4.3. *Let $T_0 \in \mathcal{B}(\mathcal{H})$ be a multiplicity-free operator of class C_0 and let $T = \bigoplus_{n=0}^\infty T_0$. Given M_1 and M_2 two invariant subspaces for T , we have that $M_1 \prec M_2$ if and only if $T|M_1 \sim T|M_2$ and $T_{M_2^\perp} \prec^i T_{M_1^\perp}$.*

Proof. Denote by θ the minimal function of T_0 . By assumption, we can find quasiaffinities $X : \mathcal{H} \rightarrow H(\theta)$ and $Y : H(\theta) \rightarrow \mathcal{H}$ with the property that $XT = S(\theta)X$, $YS(\theta) = TY$. Define $A = \bigoplus_{n=0}^\infty X$ and $B = \bigoplus_{n=0}^\infty Y$, which are quasiaffinities intertwining T with its Jordan model $\bigoplus_{n=0}^\infty S(\theta)$. By Theorem 4.2, we need only show that $BA \in \text{Alg Lat}(T)$.

Since T_0 is multiplicity-free, Proposition 2.3 implies that $\{T_0\}'$ is commutative, and thus $\{T_0\}'' = \{T_0\}'$. Therefore, $YX \in \{T_0\}''$ and $BA = \bigoplus_{n=0}^\infty YX$ then clearly belongs to $\{T\}''$ since

$$\{T\}' = \{(C_{nm})_{n,m=0}^\infty : C_{nm} \in \{T_0\}'\}.$$

By Theorem 2.4, we find that $BA \in \text{Alg Lat}(T)$ which completes the proof. \square

5. ACKNOWLEDGEMENTS

The author was supported by a NSERC PGS grant.

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